$D_{n+1}^{(2)}$ Reflection K-Matrices

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Abstract

We investigate the possible regular solutions of the boundary Yang-Baxter equation for the vertex models associated to the $D_{n+1}^{(2)}$ affine Lie algebra. We have classified them in terms of three types of K-matrices. The first one have $n+2$ free parameters and all the matrix elements are non-null. The second solution is given by a block diagonal matrix with just one free parameter. It turns out that for n even there exists a third class of K-matrix without free parameter.

February 8, 2008

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1 Introduction

Recently there has been a lot of efforts in introducing boundaries into integrable systems for possible applications to condensed matter physics and statistical system with non-periodic boundary conditions. The bulk Boltzmann weights of an exactly solvable lattice system are usually the non-null matrix elements of a R-matrix $R(u)$ which satisfies the Yang-Baxter equation. The boundaries entail new physical quantities called reflection matrices which depend on the boundary properties.

By considering systems on a finite interval with independent boundary conditions at each end, we have to introduce reflection matrices to describe such boundary conditions. Integrable models with boundaries can be constructed out of a pair of reflection K-matrices $K_{\pm}(u)$ in addition to the R-matrix. $K_{+}(u)$ and $K_{-}(u)$ describe the effects of the presence of boundaries at the left and the right ends, respectively.

Integrability of open chains in the framework of the quantum inverse scattering method was pioneered by Sklyanin. In reference [1], Sklyanin has used his formalism to solve, via algebraic Bethe ansatz, the open spin-1/2 chain with diagonal boundary terms. This model had already been solved via coordinate Bethe ansatz by Alcaraz *et al* [2].

The Sklyanin original formalism was extended to more general systems by Mezincescu and Nepomechie in [3], where is assumed that for a regular R-matrix satisfying the following properties

$$\begin{aligned} PT-symmetry &: \quad P_{12}R_{12}(u)P_{12} = R_{21}(u), \\ unitarity &: \quad R_{12}(u)R_{21}(-u) = 1, \\ crossing\ unitarity &: \quad R_{12}(u) = (U \otimes 1)R_{12}^{t_2}(-u-\rho)(U \otimes 1)^{-1}, \end{aligned} \tag{1.1}$$

one can derive an integrable open chain Hamiltonian

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2}\left(\frac{dK_{-}(u)}{du}\bigg|_{u=0} \otimes 1\right) + \frac{\mathrm{tr}_0 \overset{0}{K}_{+}(0) H_{N,0}}{\mathrm{tr} K_{+}(0)}, \tag{1.2}$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, etc. $P = R(0)$ is the permutation matrix and the two-site bulk Hamiltonian $H_{k,k+1}$ is given by $H_{k,k+1} = P_{k,k+1}\left[dR_{k,k+1}(u)/du\right]_{u=0}$.

The matrix $K_{-}(u)$ satisfies the right boundary Yang-Baxter equation, also known as the reflection equation (RE)

$$R_{12}(u-v)\overset{1}{K}_{-}(u)R_{21}(u+v)\overset{2}{K}_{-}(v) = \overset{2}{K}_{-}(v)R_{12}(u+v)\overset{1}{K}_{-}(u)R_{21}(u-v) \tag{1.3}$$

which governs the integrability at boundary for a given bulk theory. Similar equation should also hold for the matrix $K_{+}(u)$ at the opposite boundary. However, for the case of models whose matrix $R(u)$ satisfies (1.1), one can show that the corresponding quantity

$$K_{+}(u) = K_{-}^{\mathrm{t}}(-u-\rho)M, \qquad M = U^{\mathrm{t}}U = M^{\mathrm{t}}, \tag{1.4}$$

satisfy the left RE. Here ρ is a crossing parameter and U is a crossing matrix both being specific to each model [4, 5]. t_i stands for the transposition taken in the *i-th* space and tr_0 is the trace taken in the auxiliary space.

Due to the significance of the RE in the construction of integrable models with open boundaries, a lot of work has been directed to the study [6, 7, 8, 9] and classification [10, 11, 12] of K-matrices.

In spite of all these works, there is an interesting vertex model based on the non-exceptional $D_{n+1}^{(2)}$ Lie algebra for which, until recently, little was known about the solution of the corresponding RE [13].

While the investigation of particular solutions has been made to a number of lattice models, Batchelor *at al* [10] have derived diagonal solutions of the RE for face and vertex models associated with several affine Lie algebras, the classification of all possible K-matrices has been a harder problem. However, recently we have proposed a method which allowed to classify all possible K-matrices solutions for 19-vertex models[11].

In this paper we would to demonstrate that this technique is indeed much more general and can be used, in principle, to study all vertex models based on the Lie algebras [4, 5]. In order to show that we choice the $D_{n+1}^{(2)}$ vertex model since it is seen as the more difficult case among the non-exceptional cases, as argued recently by Martins and Guan [13].

We have organized this paper as follows. In section 2 we present the $D_{n+1}^{(2)}$reflection equations and in section 3 their solutions are derived and classified in three types. The last section is reserved for the conclusion. The $D_{2}^{(2)}$ type-I solution is presented in appendix.

2 The $\mathbf{D}^{(2)}_{n+1}$ reflection equations

The R-matrix for the vertex models associated to the $D^{(2)}_{n+1}$ affine Lie algebra as presented by Jimbo in [14] has the form

$$\begin{aligned} R \;=\; & \sum_{i,j\neq n+1,n+2} a_{ij}\; E_{ij}\otimes E_{i'j'} + a_1 \sum_{i\neq n+1,n+2} E_{ii}\otimes E_{ii} + a_2 \sum_{\substack{i\neq j,j' \\ i \text{ or } j\neq n+1,n+2}} E_{ii}\otimes E_{jj} \\ & + a_3 \sum_{\substack{i<j,i\neq j' \\ i,j\neq n+1,n+2}} E_{ij}\otimes E_{ji} + a_4 \sum_{\substack{i>j,i\neq j' \\ i,j\neq n+1,n+2}} E_{ij}\otimes E_{ji} \\ & + a_5 \sum_{\substack{i<n+1 \\ j\neq n+1,n+2}} (E_{ij}\otimes E_{ji} + E_{j'i'}\otimes E_{i'j'}) + a_6 \sum_{\substack{i>n+2 \\ j\neq n+1,n+2}} (E_{ij}\otimes E_{ji} + E_{j'i'}\otimes E_{i'j'}) \\ & + a_7 \sum_{\substack{i<n+1 \\ j\neq n+1,n+2}} (E_{ij}\otimes E_{j'i} + E_{j'i'}\otimes E_{i'j}) + a_8 \sum_{\substack{i>n+2 \\ j\neq n+1,n+2}} (E_{ij}\otimes E_{j'i} + E_{j'i'}\otimes E_{i'j}) \\ & + \sum_{\substack{i\neq n+1,n+2 \\ j=n+1,n+2}} \left(b_i^+\,[E_{ij}\otimes E_{i'j'} + E_{j'i'}\otimes E_{ji}] + b_i^-\,[E_{ij}\otimes E_{i'j} + E_{ji'}\otimes E_{ji}]\right) \\ & + \sum_{i=n+1,n+2} \left[c^+ E_{ii}\otimes E_{i'i'} + c^- E_{ii}\otimes E_{ii} + d^+ E_{ii'}\otimes E_{i'i} + d^- E_{ii'}\otimes E_{ii'}\right] \end{aligned} \tag{2.1}$$

where E_{ii} denotes the elementary $2n+2$ by $2n+2$ matrices $(\delta_{ia}\delta_{ib})$ and the Boltzmann weights with functional dependence on the spectral parameter u are given by

$$\begin{aligned} a_1 &= (\mathrm{e}^{2u}-q^2)(\mathrm{e}^{2u}-q^{2n}), & a_2 &= q(\mathrm{e}^{2u}-1)(\mathrm{e}^{2u}-q^{2n}), \\ a_3 &= -(q^2-1)(\mathrm{e}^{2u}-q^{2n}), & a_4 &= \mathrm{e}^{2u}a_3, \\ a_5 &= \frac{1}{2}(\mathrm{e}^{u}+1)a_3, & a_6 &= \frac{1}{2}(\mathrm{e}^{u}+1)\mathrm{e}^{u}a_3, \\ a_7 &= -\frac{1}{2}(\mathrm{e}^{u}-1)a_3, & a_8 &= \frac{1}{2}(\mathrm{e}^{u}-1)\mathrm{e}^{u}a_3, \end{aligned} \tag{2.2}$$

and for $i,j\neq n+1,n+2$

$$a_{ij} = \begin{cases} (q^2\mathrm{e}^{2u}-q^{2n})(\mathrm{e}^{2u}-1) & (i=j) \\ (q^2-1)\;\left(q^{2n+\bar{i}-\bar{j}}(\mathrm{e}^{2u}-1)-\delta_{ij'}(\mathrm{e}^{2u}-q^{2n})\right) & (i<j) \\ (q^2-1)\mathrm{e}^{2u}\left(q^{\bar{i}-\bar{j}}(\mathrm{e}^{2u}-1)-\delta_{ij'}(\mathrm{e}^{2u}-q^{2n})\right) & (i>j) \end{cases} \tag{2.3}$$

$$b_i^{\pm} = \begin{cases} \pm\, q^{i-1/2}\;(q^2-1)\;(\mathrm{e}^{2u}-1)\;(\mathrm{e}^{u}\pm q^n) & (i<n+1) \\ q^{i-n-5/2}\;(q^2-1)\;(\mathrm{e}^{2u}-1)\;\mathrm{e}^{u}(\mathrm{e}^{u}\pm q^n) & (i>n+2) \end{cases} \tag{2.4}$$

$$c^{\pm} = \pm\frac{1}{2}(q^2-1)\ (q^n+1)\ \mathrm{e}^u(\mathrm{e}^u \mp 1)\ (\mathrm{e}^u \pm q^n) + q\ (\mathrm{e}^{2u}-1)\ (\mathrm{e}^{2u}-q^{2n}), \tag{2.5}$$

$$d^{\pm} = \pm\frac{1}{2}(q^2-1)\ (q^n-1)\ \mathrm{e}^u(\mathrm{e}^u \pm 1)\ (\mathrm{e}^u \pm q^n), \tag{2.6}$$

with special attention to the notation

$$\bar{i} = \begin{cases} i+1\ , & (i < n+1) \\ n+\frac{3}{2}, & (i = n+1, n+2) \\ i-1\ , & (i > n+2) \end{cases} \qquad \text{and} \qquad i' = 2n+3-i. \tag{2.7}$$

Here $q = \mathrm{e}^{-2\eta}$ denotes an arbitrary parameter.

For each value of n the matrix $R(u)$ is regular $R(0) = f(0)P$ and satisfy PT-symmetry and unitarity

$$R_{21}(u) = P_{12}R_{12}(u)P_{12}, \qquad R_{12}(u)R_{21}(-u) = f(u)f(-u)\mathbf{1}, \tag{2.8}$$

where

$$f(u) = (\mathrm{e}^{2u}-q^2)(\mathrm{e}^{2u}-q^{2n}). \tag{2.9}$$

After we normalize the Boltzmann weights by a factor $q^{n+1}\mathrm{e}^{2u}$, the crossing-unitarity symmetry

$$R_{12}(u) = (U\otimes 1)R_{12}^{\mathrm{t}_2}(-u-\rho)(U\otimes 1)^{-1}, \tag{2.10}$$

holds with the crossing matrices U and crossing parameters ρ, respectively given by

$$U_{ij} = \delta_{i'j}\ q^{(\bar{i}-\bar{j})/2}, \qquad \rho = -n\ln q = 2n\eta \tag{2.11}$$

Regular solutions of the RE mean that the matrix $K_{-}(u)$ in the form

$$K_{-}(u) = \sum_{i,j=1}^{2n+2} k_{ij}(u)\ E_{ij} \tag{2.12}$$

satisfies the condition

$$k_{ij}(0) = \delta_{ij}, \qquad i,j = 1,2,...,2n+2 \tag{2.13}$$

Substituting (2.1) and (2.12) into (1.3), we have in both $16(n+1)^4$ functional equations for the k_{ij} elements, many of which are dependent. In order to solve them, we shall proceed in the following way. First we consider the (i,j) component of the

matrix equation (1.3). By differentiating it with respect to v and taking $v = 0$, we will get algebraic equations involving the single variable u and $4(n+1)^2$ parameters

$$\beta_{ij} = \frac{dk_{ij}(v)}{dv}|_{v=0} \qquad i,j = 1,2,...,2n+2 \tag{2.14}$$

Second, these algebraic equations are denoted by $E[i,j] = 0$ and collected into blocks $B[i,j]$, $i = 1,...,2(n+1)^2$ and $j = i, i+1, ..., (2n+2)^2 - i$, defined by

$$B[i,j] = \begin{cases} E[i,j] = 0, \ E[j,i] = 0, \\ E[4(n+1)^2 + 1 - i, 4(n+1)^2 + 1 - j] = 0, \\ E[4(n+1)^2 + 1 - j, 4(n+1)^2 + 1 - i] = 0. \end{cases} \tag{2.15}$$

For a given block $B[i,j]$, the equation $E[4(n+1)^2 + 1 - i, 4(n+1)^2 + 1 - j] = 0$ can be obtained from the equation $E[i,j] = 0$ by interchanging

$$\begin{array}{llll} k_{ij} \longleftrightarrow k_{i'j'}, & \beta_{ij} \longleftrightarrow \beta_{i'j'}, & b_i^{\pm} \longleftrightarrow b_{i'}^{\pm} \ , & a_{ij} \leftrightarrow a_{i'j'}, \\ a_3 \longleftrightarrow a_4, & a_6 \longleftrightarrow a_6, & a_7 \longleftrightarrow a_8 \ . & \end{array} \tag{2.16}$$

and the equation $E[j,i] = 0$ is obtained from the equation $E[i,j] = 0$ by the interchanging

$$k_{ij} \longleftrightarrow k_{ji}, \quad \beta_{ij} \longleftrightarrow \beta_{ji}, \quad a_{ij} \longleftrightarrow a_{j'i'} \ . \tag{2.17}$$

In this way, we can control all equations and a particular solution is simultaneously connected with at least four equations.

Since the R-matrix (2.1) satisfies unitarity, P and T invariances and crossing symmetry, the matrix $K_+(u)$ is obtained using (1.4) with the following M-matrix

$$M_{ij} = \delta_{ij} \ q^{2n+3-2\bar{i}}, \qquad i,j = 1,2,...,2n+2 \tag{2.18}$$

3 The $D_{n+1}^{(2)}$ K-matrix solutions

Analyzing the $D_{n+1}^{(2)}$ RE one can see that they possess a special structure. A lot of equations exist involving only the elements out of a block diagonal structure which consist of the diagonal elements k_{ii} plus the central elements of the secondary diagonal, $k_{n+1,n+2}$ and $k_{n+2,n+1}$. This block diagonal structure commutes with n distinct $U(1)$ symmetries, the minimal symmetry of the R-matrix [13]. Through this structure we can identify two types of solutions: the type-I solution which is a general K-matrix solution with all matrix elements non-null and the type-II solution for which the K-matrix has this block diagonal structure. Moreover, for the models with a even number of $U(1)$

conserved charge, we can find a third type of solution which manifests the $U(1)\otimes U(1)$ symmetries. Therefore for n even we can find the type-III solution which is a diagonal K-matrix. These solutions are to each other unyielding.

Having identified these possibilities we may proceed in order to find explicitly the regular solutions. We start looking for the type-I solution.

3.1 The type-I solution

The simplest RE are those involving only the elements of the secondary diagonal. We chose to express their solutions in terms of the element $k_{1,2n+2}$:

$$k_{ii'} = \frac{\beta_{ii'}}{\beta_{1,2n+2}} k_{1,2n+2}, \qquad i \neq \{n+1, n+2\}. \tag{3.1}$$

From the collections $\{B[i,j]\}$, $i = 1, 2, ..., n-1$ one can see that the RE of the last blocks of each collection are simple and we can easily solve them expressing the elements k_{ij} with $j \neq i'$ in function of $k_{1,2n+2}$

$$k_{ij} \quad = \quad \begin{cases} F_{ij} \left(\beta_{ij} a_3 a_{ii} - \beta_{j'i'} a_2 a_{ij'}\right) k_{1,2n+2}, & (i < j') \\ F_{ij} \left(\beta_{ij} a_4 a_{ii} - \beta_{j'i'} a_2 a_{ij'}\right) k_{1,2n+2}, & (i > j') \end{cases} \tag{3.2}$$

where

$$F_{ij} = \frac{a_1 a_{ii} - a_2^2}{\beta_{1,2n+2} \left(a_{ii}^2 a_3 a_4 - a_2^2 a_{ij'} a_{j'i}\right)}. \tag{3.3}$$

Moreover, for $j = n+1, n+2$ with $i \neq j, j'$ we have

$$k_{ij} \quad = \quad \begin{cases} \Delta_i \left[a_{ii} \left(\beta_{ij} a_5 + \beta_{ij'} a_7\right) - a_2 \left(\beta_{j'i'} b_i^+ + \beta_{ji'} b_i^-\right)\right] k_{1,2n+2}, & (i' > j) \\ \Delta_i \left[a_{ii} \left(\beta_{ij} a_6 + \beta_{ij'} a_8\right) - a_2 \left(\beta_{j'i'} b_i^+ + \beta_{ji'} b_i^-\right)\right] k_{1,2n+2}, & (i' < j) \end{cases} \tag{3.4}$$

and for $i = n+1, n+2$ with $j \neq i, i'$ we get

$$k_{ij} \quad = \quad \begin{cases} \Delta_j \left[a_{jj} \left(\beta_{ij} a_5 + \beta_{i'j} a_7\right) - a_2 \left(\beta_{j'i'} b_j^+ + \beta_{j'i} b_j^-\right)\right] k_{1,2n+2}, & (i' > j) \\ \Delta_j \left[a_{jj} \left(\beta_{ij} a_6 + \beta_{i'j} a_8\right) - a_2 \left(\beta_{j'i'} b_j^+ + \beta_{j'i} b_j^-\right)\right] k_{1,2n+2}, & (i' < j) \end{cases} \tag{3.5}$$

where

$$\Delta_l = \frac{a_1 a_{ll} - a_2^2}{\beta_{1,2n+2} \left(a_{ll}^2 (a_6 + a_8)(a_5 + a_7) - a_2^2 (b_l^+ + b_l^-)(b_{l'}^+ + b_{l'}^-)\right)} \tag{3.6}$$

Here we observe that for the $D_2^{(2)}$ model, $\digamma_{ij} = 0$ and $\Delta_l = \frac{0}{0}$. However, through an appropriate choice of Δ_l we can include the case $n = 1$ in our discussion (see appendix A).

Substituting these expressions in the remaining RE (in fact, it is enough to consider the equations of the collections $\{B[1,j]\}$ and $\{B[2,j]\}$), and looking at the equations of the type

$$G(u)k_{1,2n+2}(u) = 0 \tag{3.7}$$

where $G(u) = \sum_k f_k(\{\beta_{ij}\})\mathrm{e}^{ku}$. The constraint equations $f_k(\{\beta_{ij}\}) \equiv 0, \forall k$, can be solved in terms of the $2n+2$ parameters which allow us to find all k_{ij} elements out of the block-diagonal structure in terms of the $k_{1,2n+2}$.

Of course the expressions for k_{ij} will depend on our choice of these parameters. After some attempts we concluded the choice β_{12}, β_{13}, ..., $\beta_{1,2n+2}$ and β_{21} as the most appropriate for our purpose.

Taking into account the fixed parameters and the Boltzmann weights defined above we can rewrite these k_{ij} matrix elements for $n \neq 1$ in the following way:

The elements in the secondary diagonal of $K_-(u)$ are given by

$$k_{i,i'}(u) = q^{\bar{i}-2n}\left(\frac{q^{n-1}+1}{q+1}\right)^2\left(\frac{\beta_{1,i'}}{\beta_{1,2n+2}}\right)^2 k_{1,2n+2}(u), \qquad (i \neq 1, 2n+2) \tag{3.8}$$

and

$$k_{2n+2,1}(u) = q^{2n-3}\left(\frac{\beta_{21}}{\beta_{1,2n+1}}\right)^2 k_{1,2n+2}(u). \tag{3.9}$$

The elements of the first row and the elements of the first column are, respectively given by

$$k_{1,j}(u) = \left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\Gamma_{1,j}(u)}{\beta_{1,2n+2}}\right) k_{1,2n+2}(u), \qquad (j \neq 2n+2) \tag{3.10}$$

$$k_{i,1}(u) = q^{\bar{i}-3}\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\Gamma_{1,i'}(u)}{\beta_{1,2n+2}}\right)\left(\frac{\beta_{21}}{\beta_{1,2n+1}}\right) k_{1,2n+2}(u), \qquad (i \neq 2n+2) \tag{3.11}$$

while for the last column and for the last row, we have

$$k_{i,2n+2}(u) = q^{\bar{i}-n-2}\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\Pi_{1,i'}(u)}{\beta_{1,2n+2}}\right)\mathrm{e}^{2u} k_{1,2n+2}(u), \qquad (i \neq 1) \tag{3.12}$$

$$k_{2n+2,j}(u) = q^{n-2}\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\beta_{21}}{\beta_{1,2n+1}}\right)\left(\frac{\Pi_{1,j}(u)}{\beta_{1,2n+2}}\right)\mathrm{e}^{2u}k_{1,2n+2}(u), \qquad (j \neq 1) \tag{3.13}$$

The remaining matrix elements are given by

$$k_{ij}(u) = q^{\bar{i}-n-1}\left(\frac{q^{n-1}+1}{q+1}\right)\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\Gamma_{1,i'}(u)}{\beta_{1,2n+2}}\right)\left(\frac{\Gamma_{1,j}(u)}{\beta_{1,2n+2}}\right)k_{1,2n+2}(u), \tag{3.14}$$

for $i' > j$ and

$$k_{ij}(u) = q^{\bar{i}-2n-1}\left(\frac{q^{n-1}+1}{q+1}\right)\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\left(\frac{\Pi_{1,i'}(u)}{\beta_{1,2n+2}}\right)\left(\frac{\Pi_{1,j}(u)}{\beta_{1,2n+2}}\right)\mathrm{e}^{2u}k_{1,2n+2}(u) \tag{3.15}$$

for $i' < j$.

In these expressions we are using a compact notation defined by

$$\Gamma_{1,a}(u) = \begin{cases} \beta_{1,a}, & (a \neq n+1, n+2) \\ \frac{1}{2}\left(e^{u}\beta_{-}+\beta_{+}\right), & (a = n+1) \\ \frac{1}{2}\left(-e^{u}\beta_{-}+\beta_{+}\right), & (a = n+2) \end{cases} \tag{3.16}$$

and

$$\Pi_{1,a}(u) = \begin{cases} \beta_{1,a}, & (a \neq n+1, n+2) \\ \frac{1}{2}\left(q^{n}e^{-u}\beta_{-}+\beta_{+}\right), & (a = n+1) \\ \frac{1}{2}\left(-q^{n}e^{-u}\beta_{-}+\beta_{+}\right), & (a = n+2) \end{cases} \tag{3.17}$$

where $\beta_{\pm} = \beta_{1,n+1} \pm \beta_{1,n+2}$. To include the case $n = 1$ we will make some modifications in these expressions (see appendix A).

At this point we find the $2n(2n+3)$ matrix elements in terms of the $2n+2$ parameters. However, we still need to find the $2n+4$ matrix elements that belong to the block diagonal.

The block diagonal structure has the form

$$\mathrm{diag}\left(k_{11}, k_{22}, \cdots, k_{nn}, \mathcal{B}, k_{n+3,n+3}, \cdots, k_{2n+2,2n+2}\right), \tag{3.18}$$

where $\mathcal{B}$ contains the central elements

$$\mathcal{B} = \begin{pmatrix} k_{n+1,n+1} & k_{n+1,n+2} \\ k_{n+2,n+1} & k_{n+2,n+2} \end{pmatrix} \tag{3.19}$$

Here the situation is a little bit different. It is very cumbersome to write these matrix elements in terms of the Boltzmann weights. But, after some algebraic manipulations, it is possible to see that the diagonal elements satisfy two distinct recurrence relations

$$k_{ii} = \begin{cases} k_{11} - \left(\frac{q^{n-1}+1}{e^{2u}+q^{n-1}}\right)\left(\frac{\beta_{11}-\beta_{ii}}{\beta_{1,2n+2}}\right) k_{1,2n+2}, & (i < n+1) \\ k_{n+3,n+3} - \left(\frac{q^{n-1}+1}{e^{2u}+q^{n-1}}\right)\left(\frac{\beta_{n+3,n+3}-\beta_{ii}}{\beta_{1,2n+2}}\right) e^{2u} k_{1,2n+2}, & (i > n+2) \end{cases} \tag{3.20}$$

Substituting (3.20) into the RE we can find k_{11} and $k_{n+3,n+3}$ and consequently, all $k_{ii} \notin \mathcal{B}$ will be known after finding the $2n$ parameters β_{ii}.

The solution of this problem depends on the parity of n. Besides, in this stage, all remaining parameters β_{ij}, including those associated with the central elements, are fixed in terms of $n+3$ parameters:

For n odd the solution is

$$\begin{aligned} k_{11}(u) &= \left\{\left(\frac{q^{n-1}+1}{q+1}\right)\frac{(q+1)(e^{2u}+1)(q^n\beta_-^2-\beta_+^2)+2(e^{2u}-q^n)(q^n\beta_-^2+\beta_+^2)}{8\beta_{1,2n+2}^2\ q^{n-1/2}(e^{2u}+1)}\right. \\ &\left.+\frac{2q(q^{n-1}-1)+(q-1)(e^{2u}+1)}{\beta_{1,2n+2}(q^n-1)(e^{2u}-1)}\right\}\left(\frac{q^{n-1}+1}{e^{2u}+q^{n-1}}\right)k_{1,2n+2}(u) \end{aligned} \tag{3.21}$$

$$\begin{aligned} k_{n+3,n+3}(u) &= \left\{\left(\frac{q^{n-1}+1}{q+1}\right)\frac{(q+1)(e^{2u}+1)(q^n\beta_-^2-\beta_+^2)-2(e^{2u}-q^n)(q^n\beta_-^2+\beta_+^2)}{8\beta_{1,2n+2}^2\ q^{n-1/2}(e^{2u}+1)}\right. \\ &\left.+\frac{2q(q^{n-1}-1)+(q-1)(e^{2u}+1)}{\beta_{1,2n+2}(q^n-1)(e^{2u}-1)}\right\}\left(\frac{q^{n-1}+1}{e^{2u}+q^{n-1}}\right)e^{2u}k_{1,2n+2}(u) \end{aligned} \tag{3.22}$$

The parameters β_{ii} , $i \neq n+1, n+2$ are fixed by the following recurrence relations

$$\beta_{i+1,i+1} = \begin{cases} \beta_{ii} + (-q)^{i-1}\Theta_{\rm odd} & (i < n) \\ \beta_{ii} + (-q)^{i-n-3}\Theta_{\rm odd} & (i > n+2) \end{cases} \tag{3.23}$$

with

$$\beta_{n+3,n+3} = \beta_{11} + 2 + \left(\frac{q^{n-1}+1}{q+1}\right)\frac{(q^n-1)(q^n\beta_-^2+\beta_+^2)}{4q^{n-1/2}\beta_{1,2n+2}} \tag{3.24}$$

and

$$\Theta_{\rm odd} = -\frac{(q+1)^2}{q^n-1} - \frac{(q+1)(q^{n-1}+1)(\beta_-^2q^n-\beta_+^2)}{8q^{n-1/2}\beta_{1,2n+2}} \tag{3.25}$$

Finally, we can solve the last RE to find the central elements. The solution is

$$
\begin{aligned}
k_{n+1,n+1}(u) &= -\left\{\left(\frac{q^{n-1}+1}{q+1}\right)\frac{q^n\beta_-^2-\beta_+^2}{8q^{n-1/2}}(\mathrm{e}^{2u}+q^n)\right.\\
&\left.+\beta_{1,2n+2}\frac{(\mathrm{e}^{2u}+1)(\mathrm{e}^{2u}-q^n)}{(\mathrm{e}^{2u}-1)(q^n-1)}\right\}\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\frac{k_{1,2n+2}(u)}{\beta_{1.2n+2}^2},\\
k_{n+2,n+2}(u) &= k_{n+1,n+1}(u)
\end{aligned}
\tag{3.26}
$$

$$
k_{n+1,n+2}(u) = \frac{(q^{n-1}+1)^2\left[(q^n+1)(q^n\beta_-^2+\beta_+^2)\mathrm{e}^u-2q^n\beta_-\beta_+(\mathrm{e}^{2u}+1)\right]}{4q^{n-1/2}(q+1)(\mathrm{e}^{2u}+1)(\mathrm{e}^{2u}+q^{n-1})}\frac{\mathrm{e}^u k_{1,2n+2}(u)}{\beta_{1,2n+2}^2},
\tag{3.27}
$$

$$
k_{n+2,n+1}(u) = \frac{(q^{n-1}+1)^2\left[(q^n+1)(q^n\beta_-^2+\beta_+^2)\mathrm{e}^u+2q^n\beta_-\beta_+(\mathrm{e}^{2u}+1)\right]}{4q^{n-1/2}(q+1)(\mathrm{e}^{2u}+1)(\mathrm{e}^{2u}+q^{n-1})}\frac{\mathrm{e}^u k_{1,2n+2}(u)}{\beta_{1,2n+2}^2}.
\tag{3.28}
$$

Moreover, there are n parameters which have been fixed by the RE

$$
\beta_{21} = \frac{1}{q^{2n-3}}\left(\frac{q^{n-1}+1}{q+1}\right)^2\frac{\beta_{1,n}\beta_{1,n+3}\beta_{1,2n+1}}{\beta_{1,2n+2}^2}, \qquad (n\neq 1)
\tag{3.29}
$$

$$
\beta_{1,n} = \frac{\left[(q^n-1)(q^{n-1}+1)(q^n\beta_-^2-\beta_+^2)+8q^{n-1/2}(q+1)\beta_{1,2n+2}\right](q+1)}{8\sqrt{q}(q^n-1)(q^{n-1}+1)\beta_{1,n+3}}, \qquad (n\neq 1)
\tag{3.30}
$$

$$
\beta_{1j} = (-)^{j-1}\frac{\beta_{1,n}\beta_{1,n+3}}{\beta_{1,2n+3-j}}, \qquad j=2,3,\cdots,n-1.
\tag{3.31}
$$

The final result is a general solution with $n+3$ free parameters β_{11}, $\beta_{1,n+1}$, $\beta_{1,n+2}$, ..., $\beta_{1,2n+2}$.

By the choose

$$
k_{1,2n+2}(u) = \frac{1}{2}\beta_{1,2n+2}\left(\mathrm{e}^{2u}-1\right),
\tag{3.32}
$$

one can , for instance, to fix the parameter β_{11} using the regular condition (2.13) and in that way to end with a $(n+2)$-parameter solution for $D_{n+1}^{(2)}$ models with n odd.

The corresponding matrix $K_+(u)$ is obtained used (1.4) and in this case we have $\mathrm{Tr}(K_+(0))\neq 0$. Therefore, the equation (1.2) gives the corresponding integrable open chain Hamiltonians, where $H_{k,k+1}$ are the tensors $s_{k,k+1}(q)$ derived by Jimbo [14].

Now we turn to describe the even solution. In the n-even case we find the following expressions for k_{11} and $k_{n+3,n+3}$, respectively

$$
\begin{aligned}
k_{11}(u) &= \frac{k_{1,2n+2}(u)}{2q^{n-1/2}\beta_{1,2n+2}^2(\mathrm{e}^{4u}-1)(\mathrm{e}^{2u}+q^{n-1})}\big\{(q+1)(\mathrm{e}^{2u}-q^n)(q^n\beta_-^2\mathrm{e}^{2u}-\beta_+^2) \\
&\quad +2\sqrt{q}\;\beta_{1,n}\beta_{1,n+3}(\mathrm{e}^{2u}-1)\left[2(\mathrm{e}^{2u}-q^n)-(q+1)(\mathrm{e}^{2u}+1)\right]\big\}
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
k_{n+3,n+3}(u) &= \frac{\mathrm{e}^{2u}k_{1,2n+2}(u)}{2q^{n-1/2}\beta_{1,2n+2}^2(\mathrm{e}^{4u}-1)(\mathrm{e}^{2u}+q^{n-1})}\big\{(q+1)(\mathrm{e}^{2u}-q^n)(q^n\beta_-^2-\beta_+^2\mathrm{e}^{2u}) \\
&\quad -2\sqrt{q}\;\beta_{1,n}\beta_{1,n+3}(\mathrm{e}^{2u}-1)\left[2(\mathrm{e}^{2u}-q^n)-(q+1)(\mathrm{e}^{2u}+1)\right]\big\}
\end{aligned} \tag{3.34}
$$

The central elements are given by

$$
k_{n+1,n+1}(u)=k_{n+2,n+2}(u)=2\left(\frac{\mathrm{e}^{2u}}{\mathrm{e}^{2u}-1}\right)\left(\frac{q^{n-1}+1}{\mathrm{e}^{2u}+q^{n-1}}\right)\frac{k_{1,2n+2}(u)}{\beta_{1,2n+2}} \tag{3.35}
$$

$$
\begin{aligned}
k_{n+1,n+2}(u) &= \frac{k_{1,2n+2}(u)}{4q^{n-1/2}\beta_{1,2n+2}^2(\mathrm{e}^{2u}+q^{n-1})(\mathrm{e}^{2u}+1)}\left(\frac{q^{n-1}+1}{q+1}\right)^2 \\
&\quad \Big\{\left(q^n\beta_-^2+\beta_+^2\right)^2(q+1)(q^n+1)\mathrm{e}^{2u}-2\beta_-\beta_+(q+1)q^n\mathrm{e}^{u}(\mathrm{e}^{2u}+1) \\
&\quad -4\sqrt{q}\beta_{1,n}\beta_{1,n+3}(\mathrm{e}^{2u}-q^n)(\mathrm{e}^{2u}-1)\Big\}
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
k_{n+2,n+1}(u) &= \frac{k_{1,2n+2}(u)}{4q^{n-1/2}\beta_{1,2n+2}^2(\mathrm{e}^{2u}+q^{n-1})(\mathrm{e}^{2u}+1)}\left(\frac{q^{n-1}+1}{q+1}\right)^2 \\
&\quad \Big\{\left(q^n\beta_-^2+\beta_+^2\right)^2(q+1)(q^n+1)\mathrm{e}^{2u}+2\beta_-\beta_+(q+1)q^n\mathrm{e}^{u}(\mathrm{e}^{2u}+1) \\
&\quad -4\sqrt{q}\beta_{1,n}\beta_{1,n+3}(\mathrm{e}^{2u}-q^n)(\mathrm{e}^{2u}-1)\Big\}
\end{aligned} \tag{3.37}
$$

The β_{ii} parameters are fixed by the following recurrence relations

$$
\beta_{i+1,i+1}=\begin{cases} \beta_{ii}+(-q)^{i-1}\Theta_{\text{even}}, & (i<n) \\ \beta_{ii}-(-q)^{i-n-3}\Theta_{\text{even}}, & (i>n+2) \end{cases} \tag{3.38}
$$

with

$$
\beta_{n+3,n+3}=\beta_{11}+2-\frac{2(q^n-1)(q+1)(q^n\beta_-^2+\beta_+^2)+8\beta_{1,n}\beta_{1,n+3}q^{3/2}(q^{n-1}+1)}{(q+1)(q^n-1)(q^n\beta_-^2-\beta_+^2)} \tag{3.39}
$$

and

$$\Theta_{\text{even}} = -8\frac{\sqrt{q}(q+1)}{q^n - 1}\frac{\beta_{1,n}\beta_{1,n+3}}{\beta_-^2 q^n - \beta_+^2} \tag{3.40}$$

Now the n fixed parameters are

$$\beta_{21} = -q^{3-2n}\left(\frac{q^{n-1}+1}{q+1}\right)\beta_{1,2n+1}\frac{\beta_{1,n}\beta_{1,n+3}}{\beta_{1,2n+2}^2} \tag{3.41}$$

$$\beta_{1,2n+2} = -\frac{1}{8}\left(\frac{q^n - 1}{q^{n-1/2}}\right)\left(\frac{q^{n-1}+1}{q+1}\right)(q^n\beta_-^2 - \beta_+^2) \tag{3.42}$$

$$\beta_{1,j} = (-)^{j-1}\frac{\beta_{1,n}\beta_{1,n+3}}{\beta_{1,2n+3-j}}, \qquad j = 2, 3, \cdots, n-1 \tag{3.43}$$

and the $n+3$ free parameters are β_{11},$\beta_{1,n}$,..., $\beta_{1,2n+1}$.

Again, one can fix β_{11} from the regular condition to get a general solution with $n+2$ free parameters. Here we also have $\mathrm{Tr}(K_+(0)) \neq 0$, $\forall n$ and (1.2) gives the corresponding integrable open chain Hamiltonians.

There are many particular solutions which are obtained by vanishing some of these free parameters. Despite the existence of $n+2$ free parameters in a type-I solution it seems impossible to make a reduction through the annulment of certain parameters in order to obtain a type-II or type-III solution.

3.2 The Type-II solution

As was already mentioned, the $D_{n+1}^{(2)}$ models have n distinct $U(1)$ conserved charges, and the K-matrix ansatz compatible with these symmetries is the block diagonal structure presented in the previous section.

Looking for the general solution of the corresponding RE we find that the only possible solution is obtained when the two recurrence relations (3.20) are degenerated into k_{11} and into $k_{n+3,n+3}$, respectively.

$$k_{n,n}(u) = k_{n-1,n-1}(u) = \cdots = k_{22}(u) = k_{11}(u), \tag{3.44}$$

and

$$k_{2n+2,2n+2}(u) = k_{2n+1,2n+1}(u) = \cdots = k_{n+4,n+4}(u) = k_{n+3,n+3}(u) \tag{3.45}$$

The type-II solution can be obtained by the same procedure described before and in what follows we only quote the final results.

We have found two solutions for any value of n. The first solution is given by

$$k_{11}(u)=\frac{1}{2}\frac{(\mathrm{e}^{2u}+q^{n})\left\lfloor(q^{n}-1)(\mathrm{e}^{2u}+1)-\beta_{n+1,n+2}(q^{n}+1)(\mathrm{e}^{2u}-1)\right\rfloor}{\mathrm{e}^{2u}(q^{2n}-1)} \tag{3.46}$$

$$k_{n+3,n+3}(u)=\frac{1}{2}\frac{(\mathrm{e}^{2u}+q^{n})\left\lfloor(q^{n}-1)(\mathrm{e}^{2u}+1)+\beta_{n+1,n+2}(q^{n}+1)(\mathrm{e}^{2u}-1)\right\rfloor}{(q^{2n}-1)} \tag{3.47}$$

with central elements

$$k_{n+1,n+2}(u)=k_{n+2,n+1}(u)=\frac{1}{2}\beta_{n+1,n+2}(\mathrm{e}^{2u}-1) \tag{3.48}$$

$$k_{n+1,n+1}(u)=\frac{1}{2}(\mathrm{e}^{2u}+1)\left\{1+\frac{(\mathrm{e}^{2u}-1)}{\mathrm{e}^{u}(q^{2n}-1)}\Gamma_{\pm}\right\} \tag{3.49}$$

$$k_{n+2,n+2}(u)=\frac{1}{2}(\mathrm{e}^{2u}+1)\left\{1+\frac{(\mathrm{e}^{2u}-1)}{\mathrm{e}^{u}(q^{2n}-1)}\Gamma_{\mp}\right\} \tag{3.50}$$

where

$$\begin{aligned}\Gamma_{\pm} &= \frac{1}{\Sigma_{\pm}}\left\{2q^{n}\left[(q^{n}+1)^{2}\beta_{n+1,n+2}^{2}-(q^{n}-1)^{2}\right]\right.\\ &\left.\pm\left[(q^{n}+1)^{2}\beta_{n+1,n+2}+(q^{n}-1)^{2}\right]\sqrt{q^{n}\left[(q^{n}+1)^{2}\beta_{n+1,n+2}^{2}-(q^{n}-1)^{2}\right]}\right\}\end{aligned} \tag{3.51}$$

and

$$\Sigma_{\pm}=\left[(q^{n}+1)^{2}\beta_{n+1,n+2}+(q^{n}-1)^{2}\right]\pm2\sqrt{q^{n}\left[(q^{n}+1)^{2}\beta_{n+1,n+2}^{2}-(q^{n}-1)^{2}\right]} \tag{3.52}$$

The signs $(\pm)$ and $(\pm)$ are indicating the existence of two conjugated solutions. Here we notice that these solutions degenerated into two complex diagonal solutions when $\beta_{n+1,n+2}=0$.

For the second solution we have

$$\begin{aligned}k_{11}(u) &= \frac{1}{2}\frac{(\mathrm{e}^{2u}-q^{n})}{\mathrm{e}^{u}(q^{n}-1)^{2}}\left\{(\mathrm{e}^{2u}-1)\left[(q^{n}+1)\beta_{n+1,n+2}\pm2\sqrt{q^{n}\left[\beta_{n+1,n+2}^{2}-1\right]}\right]\right.\\ &\left.-(\mathrm{e}^{2u}+1)(q^{n}-1)\right\}\end{aligned} \tag{3.53}$$

$$
\begin{aligned}
k_{n+3,n+3}(u) &= -\frac{1}{2}\frac{\mathrm{e}^{u}(\mathrm{e}^{2u}-q^{n})}{(q^{n}-1)^{2}}\left\{(\mathrm{e}^{2u}-1)\left[(q^{n}+1)\beta_{n+1,n+2}\pm 2\sqrt{q^{n}\left[\beta_{n+1,n+2}^{2}-1\right]}\right]\right. \\
&\left. + (\mathrm{e}^{2u}+1)(q^{n}-1)\right\} \qquad (3.54)
\end{aligned}
$$

with the following central elements

$$
k_{n+1,n+1}(u) = k_{n+2,n+2}(u) = \frac{1}{2}\mathrm{e}^{u}(\mathrm{e}^{2u}+1) \qquad (3.55)
$$

$$
\begin{aligned}
k_{n+1,n+2}(u) &= \frac{1}{2}\frac{(\mathrm{e}^{2u}-1)}{(q^{n}-1)^{2}}\left\{\beta_{n+1,n+2}\left[\mathrm{e}^{u}(q^{n}+1)^{2}-2q^{n}(\mathrm{e}^{2u}+1)\right]\right. \\
&\left. \mp (q^{n}+1)\sqrt{q^{n}\left[\beta_{n+1,n+2}^{2}-1\right]}(\mathrm{e}^{u}-1)^{2}\right\} \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
k_{n+2,n+1}(u) &= \frac{1}{2}\frac{(\mathrm{e}^{2u}-1)}{(q^{n}-1)^{2}}\left\{\beta_{n+1,n+2}\left[\mathrm{e}^{u}(q^{n}+1)^{2}+2q^{n}(\mathrm{e}^{2u}+1)\right]\right. \\
&\left. \pm (q^{n}+1)\sqrt{q^{n}\left[\beta_{n+1,n+2}^{2}-1\right]}(\mathrm{e}^{u}+1)^{2}\right\} \qquad (3.57)
\end{aligned}
$$

Unlike of the first solution does not exist a way of deriving a diagonal solution starting from these conjugated solutions. Moreover, when $n = 1$ we have $\mathrm{Tr}(K_{+}(0)) = 0$. Therefore in this case the corresponding boundary term in the integrable open chain Hamiltonian is not more given by (1.2) but taking into account the second order expansion of the transfer matrix in the spectral parameter u [15].

3.3 The Type-III Solution

For the models with an even number of $U(1)$ conserved charge, we look for diagonal K-matrix solutions which manifest the $U(1) \otimes U(1)$ symmetries. Here there is only one solution, namely the "almost unity" solution [13]

$$
\begin{aligned}
k_{11}(u) &= \mathrm{e}^{-2u} \\
k_{22}(u) &= k_{33}(u) = \cdots = k_{2n+1,2n+1}(u) = 1 \\
k_{2n+2,2n+2}(u) &= \mathrm{e}^{2u} \qquad (3.58)
\end{aligned}
$$

In that point we finished noticing the absence of the trivial solution $K_{-}(u) = 1$ for this system.

The type-II and type-III solutions were already obtained by Martins and Guan [13].

4 Conclusion

In this paper we have investigated the regular solutions of the reflection equations for the vertex models associated to the $D^{(2)}_{n+1}$ affine Lie algebra. After a systematic study of the functional equations we find that there are three types of solutions. We call of type-I the K-matrix solutions with $n+2$ free parameters. The type-II are block diagonal matrices with one free parameter. Finally, the type-III are diagonal matrices without free parameters which only exist for n even.

The absence of an algebraic method to classify the reflection equation solutions allows us to believe that our result can be extended in order to pick up all non-exceptional Lie algebras [4, 14], including their supersymmetric partners [5].

Acknowledgment: It is a pleasure to thank M. J. Martins for invaluable discussions. This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo–FAPESP–Brasil and by Conselho Nacional de Desenvolvimento–CNPq–Brasil.

A The $\mathrm{D}^{(2)}_2$ Reflection K-matrices

In this appendix we will present the type-I for the $D^{(2)}_2$ model. The matrix $K_{-}(u)$ has the form

$$K_{-} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \tag{A.1}$$

The elements k_{11}, k_{44} and $k_{22}=k_{33}$, k_{23} , k_{32} are read directly from the n odd solution taking $n=1$ into (3.21),(3.22) and (3.26),(3.27), (3.28), respectively:

$$\begin{aligned} k_{11}(u) &= \frac{1}{2}\frac{\left[2(\mathrm{e}^{2u}-q)(q\beta_{-}^{2}+\beta_{+}^{2})+(q+1)(\mathrm{e}^{2u}+1)(q\beta_{-}^{2}-\beta_{+}^{2})\right]k_{14}(u)}{\beta_{14}^{2}\sqrt{q}(q+1)(\mathrm{e}^{2u}+1)^{2}} \\ &\quad +2\frac{k_{14}(u)}{\beta_{14}(\mathrm{e}^{2u}-1)} \end{aligned} \tag{A.2}$$

$$\begin{aligned} k_{44}(u) &= \frac{1}{2}\frac{\left[-2(\mathrm{e}^{2u}-q)(q\beta_{-}^{2}+\beta_{+}^{2})+(q+1)(\mathrm{e}^{2u}+1)(q\beta_{-}^{2}-\beta_{+}^{2})\right]\mathrm{e}^{2u}k_{14}(u)}{\beta_{14}^{2}\sqrt{q}(q+1)(\mathrm{e}^{2u}+1)^{2}} \\ &\quad +2\frac{\mathrm{e}^{2u}k_{14}(u)}{\beta_{14}(\mathrm{e}^{2u}-1)} \end{aligned} \tag{A.3}$$

$$k_{22}(u)=k_{33}(u)=-\frac{1}{2}\left(\frac{q\beta_-^2-\beta_+^2}{\sqrt{q}(q+1)}\frac{\mathrm{e}^{2u}+q}{\mathrm{e}^{2u}+1}+\frac{4\beta_{14}(\mathrm{e}^{2u}-q)}{(q-1)(\mathrm{e}^{2u}-1)}\right)\frac{k_{14}(u)}{\beta_{14}^2} \tag{A.4}$$

$$k_{23}(u)=\frac{\mathrm{e}^{u}}{\mathrm{e}^{2u}+1}\left(\frac{q\beta_-^2+\beta_+^2}{\sqrt{q}(\mathrm{e}^{2u}+1)}\mathrm{e}^{u}-\frac{2\sqrt{q}\beta_-\beta_+}{q+1}\right)\frac{k_{14}(u)}{\beta_{14}^2} \tag{A.5}$$

$$k_{32}(u)=\frac{\mathrm{e}^{u}}{\mathrm{e}^{2u}+1}\left(\frac{q\beta_-^2+\beta_+^2}{\sqrt{q}(\mathrm{e}^{2u}+1)}\mathrm{e}^{u}+\frac{2\sqrt{q}\beta_-\beta_+}{q+1}\right)\frac{k_{14}(u)}{\beta_{14}^2} \tag{A.6}$$

where $\beta_\pm=\beta_{12}\pm\beta_{13}$.

Due to the indetermination of Δ_l (3.6) when $n=1$, we can replace Δ_l into (3.4) and (3.5) by Δ_l' defined by

$$\Delta_l\to\Delta_l'=\left(\frac{q^2-1}{q^2-e^{2u}}\right)\left(\frac{e^{2u}}{1+e^{2u}}\right)\frac{1}{\beta_{13}(b_1^++b_1^-)(b_4^++b_4^-)}. \tag{A.7}$$

This replacement allows that the equations (3.9–3.13) hold for the $D_2^{(2)}$ model up to a q-factor. The result is

$$k_{41}(u)=\frac{\beta_{21}^2}{\beta_{13}^2}k_{14}(u), \tag{A.8}$$

$$k_{12}(u)=\left(\frac{\mathrm{e}^{u}\beta_-+\beta_+}{\beta_{14}(\mathrm{e}^{2u}+1)}\right)k_{14}(u),\qquad k_{13}(u)=\left(\frac{-\mathrm{e}^{u}\beta_-+\beta_+}{\beta_{14}(\mathrm{e}^{2u}+1)}\right)k_{14}(u), \tag{A.9}$$

$$k_{21}(u)=\frac{\beta_{21}}{\beta_{13}}k_{13}(u),\qquad k_{31}(u)=\frac{\beta_{21}}{\beta_{13}}k_{12}(u), \tag{A.10}$$

$$k_{42}(u)=\frac{\beta_{21}}{\beta_{13}}k_{34}(u),\qquad k_{43}(u)=\frac{\beta_{21}}{\beta_{13}}k_{24}(u). \tag{A.11}$$

$$k_{24}(u)=\frac{1}{\sqrt{q}}\left(\frac{-q\mathrm{e}^{-u}\beta_-+\beta_+}{\beta_{14}(\mathrm{e}^{2u}+1)}\right)\mathrm{e}^{2u}k_{14}(u),\qquad k_{34}(u)=\frac{1}{\sqrt{q}}\left(\frac{q\mathrm{e}^{-u}\beta_-+\beta_+}{\beta_{14}(\mathrm{e}^{2u}+1)}\right)\mathrm{e}^{2u}k_{14}(u), \tag{A.12}$$

where β_{21} is given by

$$\beta_{21}=\frac{1}{2}\left(\frac{(q-1)(q\beta_-^2-\beta_+^2)+4\sqrt{q}(q+1)\beta_{14}}{(q^2-1)\beta_{14}^2}\right)\beta_{13}. \tag{A.13}$$

By the choice

$$k_{14}(u)=\frac{1}{2}\beta_{14}(\mathrm{e}^{2u}-1) \tag{A.14}$$

we can find β_{11}

$$\beta_{11}=-\frac{2\sqrt{q}}{q+1}\frac{\beta_{12}\beta_{13}}{\beta_{14}} \tag{A.15}$$

in order to get a regular solution with three free parameters β_{12}, β_{13} and β_{14}.

References

[1] E. K. Sklyanin, *J. Phys A: Math. Gen.* **21** (1988) 2375

[2] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Math. Phys. A: Math. Gen.* **20** (1987) 6397

[3] L. Mezincescu and R. I. Nepomechie, *J. Phys. A: Math. Gen.* **24** (1991) L17

[4] V. V. Bazhanov, *Phys. Lett.* **159B** (1985) 321; *Commun. Math. Phys.* **113** (1987) 471

[5] V. V. Bazhanov and A. Schadrikov, *Theor. Math. Phys.* **73** (1987) 402

[6] I. V. Cherednik, *Theor. Math. Phys.* **61** (1984) 977, *Int. J. Mod. Phys.* **A7**, Suppl. **1B** (1992) 707

[7] H. J. de Vega and A. González-Ruiz, *J. Phys. A: Math. Gen.* **26** (1993) L519; *J. Phys. A: Math. Gen.* **27** (1994) 6129

[8] L. Mezincescu and R. I. Nepomechie, *Int. J. Mod. Phys.* **A6** (1991) 5231

[9] T. Inami, S. Odake and Y.-Z. Zhang, *Nucl. Phys.* **B470** (1996) 419

[10] M. T. Batchelor, V. Fridkin, A. Kuniba and Y. K. Zhou, *Phys. Lett.* **B376** (1996) 266

[11] A. Lima-Santos, *Nucl. Phys.* **B558** (1999) 637

[12] Cong-xin Liu, Guo-xing Ju, Shi-kun Wang and Ke Wu, *J. Phys.* **A32** (1999) 3505

[13] M. J. Martins and X.-W. Guan, *Nucl. Phys.* **B583** (2000) 721

[14] M. Jimbo, *Commun. Math. Phys.* **102** (1986) 537

[15] R. Cuerno and A. Gonzales-Ruiz, *J. Phys. A: Math. Gen.* **26** (1993) L605